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VAUGHT'S CONJECTURE FOR SOME MEAGER GROUPS

BY

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ABSTRACT

Assume G is a superstable group of $\mathcal{M}$-rank 1 and the division ring of pseudo-endomorphisms of G is a prime field. We prove a relative Vaught's conjecture for $\mathrm{Th}(G)$. When additionally $U(G) = \omega$, this yields Vaught's conjecture for $\mathrm{Th}(G)$.

0. Introduction

Throughout the paper, T is a small superstable theory, usually with few (that is, $< 2^{\aleph_0}$) countable models, and we work within a monster model $\mathfrak{C} = \mathfrak{C}^{eq}$ of T. We use the standard model-theoretic terminology (cf. [Sh], [Ba], [Hr]). Vaught's conjecture states that if T has few countable models, then T has countably many of them. Thus far, Vaught's conjecture has been proved for the case of ω-stable T ([SHM], [Ba]) and of superstable T of finite U-rank ([Bu2]).

In [Ne2] I suggested the following approach. Suppose Q is a type-definable over $\emptyset$ subset of a countable model M of T (or even a countable union of such sets), say, $Q = \Phi(M)$. Suppose we know how to classify (up to isomorphism) sets of the form $\Phi(N)$, where N is a countable model of T. Then in order to classify all countable models of T it is enough, given Q, to see how M "envelopes" Q. In other words, it is enough to classify the countable models in $K_Q = \{N\colon \Phi(N) = Q\}$. The

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investigation of the relationship between M and Q led to several conjectures and new notions (see [Ne2, Ne3, Ne4]), among others of meager type, meager group and $\mathcal{M}$-rank. Meager types and meager groups are locally modular and share many properties of weakly minimal types and groups. The $\mathcal{M}$-rank of a type p measures the size of the set of stationarizations of p. The $\mathcal{M}$-rank of a group measures the set of generic types of this group. It is proved in [Ne4], [Ne6] that if T has few countable models then $\mathcal{M}$-rank is finite.

Meager types and meager groups occur quite naturally in superstable structures: each properly weakly minimal type and properly weakly minimal group are meager. Also, if T is 1-based then any regular type is either meager or non-orthogonal to a strongly regular type ([Ne6]). $\mathcal{M}$-rank of a weakly minimal type and group is ≤ 1 (under the few models assumption); this follows from Saffe's condition, proved in [Ne1]. In [Pi] A. Pillay proved Vaught's conjecture for $T = \mathrm{Th}(G)$, where G is a meager group with the property that each forking extension of a generic type in G has Morley rank. From this assumption it follows that $\mathcal{M}(G) = 1$.

In this paper we investigate meager groups G of $\mathcal{M}$-rank 1, and in some cases prove Vaught's conjecture for $\mathrm{Th}(G)$.

By [Ne6], if T is 1-based and G is a meager group 0-definable in $\mathfrak{C}$, then for some definable non-generic group $H < G$, G/H is meager, non-orthogonal to G, and has $\mathcal{M}$-rank 1. This justifies (at least for 1-based T) our interest in meager groups of $\mathcal{M}$-rank 1.

Supose $T = \mathrm{Th}(G)$, where G is a countable meager group of $\mathcal{M}$-rank 1, and we want to prove Vaught's conjecture for $\mathrm{Th}(G)$. The most natural set to distinguish in G is the set Q of non-generic elements of G. Notice that Q is a countable union of 0-definable subsets of G. In this paper we try to prove Vaught's conjecture for $\mathrm{Th}(G)$ relative to Q, that is, to classify models in K_Q, and to prove that there are countably many of them. Here we succeed in doing so under an additional assumption regarding the geometry of G, namely that the ring of pseudo-endomorphisms of G is a prime field. Clearly, if $U(G) = \omega$ then by [Bu2] we have Vaught's conjecture for $\mathrm{Th}(Q)$, hence our result gives Vaught's conjecture for $\mathrm{Th}(G)$ (under the above assumptions on G). The proof is an application of the techniques of meager types, calculus of traces of types and Q-isolation developed in [Ne2, Ne3, Ne4, Ne5, Ne6].

1. Preliminaries

Here we recall some notions introduced in some earlier papers. In this paper Φ will be a countable disjunction of formulas over $\emptyset$ (but most of the proofs work also for the case when formulas are replaced by types over $\emptyset$) and $Q = \Phi(M^*)$ for some countable model M^* of T. Proving Vaught's conjecture for T relative to Q consists in showing (under the few models assumption) that there are only countably many countable models in $K_Q = \{M\colon \Phi(M) = Q\}$. Proving Vaught's conjecture for T relative to Φ means proving Vaught's conjecture for T relative to any countable $Q = \Phi(M^*)$.

To construct a model M in K_Q means just to find a model M containing Q and omitting the type $\Phi(x) \cup \{x \neq a, a \in Q\}$ (in fact, this is a countable family of types).

Suppose A is a countable subset of some $M \in K_Q$. We say that $p \in S(QA)$ is good if p is realized in some model in K_Q containing A. Clearly, good types form a dense G_δ-subset of $S(QA)$ [Ne2]. Let $\mathrm{Aut}(Q/A)$ be the set of automorphisms of $\mathfrak{C}$ fixing Q setwise and A pointwise. We call the orbits of the action of $\mathrm{Aut}(Q/A)$ on $S(QA)$ pseudo-types in $S(QA)$ over A. Pseudo-types are Borel, hence have the Baire property. We say that $p \in S(QA)$ is Q-isolated over A if the pseudo-type over A containing p is not meager in $S(QA)$. A model $M \in K_Q$ containing A is called Q-atomic over A if for every finite tuple $a \subset M$, $\mathrm{tp}(a/QA)$ is Q-isolated over A.

The following lemma collects the properties of these notions.

Lemma 1.1 ([Ne2]):

(1) $\mathrm{tp}(ab/QA)$ *is Q-isolated over A iff* $\mathrm{tp}(a/QA)$ *is Q-isolated over A and* $\mathrm{tp}(b/QAa)$ *is Q-isolated over Aa.*

(2) *If $p \in S(QA)$ is Q-isolated over A, then p is good.*

(3) *If A is finite and T has few countable models, then Q-isolated over A types are dense in $S(QA)$.*

(4) *If Q-isolated over A types are dense in $S_n(QA)$ for every $n < \omega$, then there is a model $M \in K_Q$ which is Q-atomic over A.*

(5) *If $M, M' \in K_Q$ are countable and Q-atomic over A, then M and M' are isomorphic over A.*

The feature that distinguishes ω-stable theories among small superstable ones is the absence of types with infinite multiplicity. In fact [M, A.16], a small superstable T is not ω-stable iff some type in T has infinite multiplicity. Hence investigating the ways in which multiplicity of a given type may be infinite seems a crucial step in an analysis of countable models of T. Now we shall introduce the tools needed to investigate this notion. One of them is a rank $\mathcal{M}$, defined on all complete types over finite sets, with values in $\mathrm{Ord} \cup \{\infty\}$. $\mathcal{M}$ measures the size of the set of stationarizations of a type.

If A is any set and $s(x)$ any type (possibly incomplete, or even just a single formula) over $\mathfrak{C}$ then $\mathrm{Tr}_A(s)$ (the trace of s over A) is the set $\{\mathrm{tp}(a/\mathrm{acl}(A)) : a \text{ realizes } s(x)\}$. Notice that $\mathrm{Tr}_A(s)$ is a closed subset of $S(\mathrm{acl}(A))$. $\mathrm{Tr}_A(a/B)$ abbreviates $\mathrm{Tr}_A(\mathrm{tp}(a/B))$, and we omit A if $A = \emptyset$.

So $\mathrm{Tr}_A(a/A)$ is just the set of stationarizations of $\mathrm{tp}(a/A)$ over A, and $\mathrm{Mlt}(\mathrm{tp}(a/A))$ is just the number of elements of $\mathrm{Tr}_A(a/A)$, which may be finite or $2^{\aleph_0}$. Thus it does not make much sense to measure $\mathrm{Tr}_A(a/A)$ by its cardinality. Instead, $\mathcal{M}$-rank measures a 'topological size' of $\mathrm{Tr}_A(a/A)$. We define $\mathcal{M}$ as the smallest function satisfying the following conditions for any a, A, B (with A, B finite).

(a) $\mathcal{M}(a/A) \geq 0$.

(b) For limit δ, $\mathcal{M}(a/A) \geq \delta$ iff $\mathcal{M}(a/A) \geq \alpha$ for every $\alpha < \delta$.

(c) $\mathcal{M}(a/A) \geq \alpha + 1$ iff for some $B \supset A$ with $a \perp B(A)$, $\mathcal{M}(a/B) \geq \alpha$ and $\mathrm{Tr}_A(a/B)$ is nowhere dense in $\mathrm{Tr}_A(a/A)$.

Clearly, $\mathcal{M}(a/A)$ depends only on $\mathrm{tp}(a/A)$, so we may define $\mathcal{M}(\mathrm{tp}(a/A))$ as $\mathcal{M}(a/A)$.

In particular $\mathcal{M}(a/A) = 0$ means that $\mathrm{tp}(a/A)$ has finite multiplicity, while $\mathcal{M}(a/A) = 1$ means that $p = \mathrm{tp}(a/A)$ has infinite multiplicity; however, for any finite set B containing A and any non-forking extension $q \in S(B)$ of p, if q is non-isolated in the set of non-forking extensions of p in $S(B)$, then q has finite multiplicity (that is, $\mathcal{M}(q) = 0$).

Regarding (c) above, it should be noted here that for any $A \subseteq B \subseteq C$ and a with $a \perp C(A)$, either $\mathrm{Tr}_A(a/C)$ is nowhere dense in $\mathrm{Tr}_A(a/B)$ or $\mathrm{Tr}_A(a/C)$ is open in $\mathrm{Tr}_A(a/B)$ [Ne5, fact 0.1]. This leads to the notion of m-independence (refining forking independence), defined by $a \overset{\mathrm{m}}{\perp} b(c)$ iff $a \perp b(c)$ and $\mathrm{Tr}_c(a/bc)$ is open in $\mathrm{Tr}_c(a/c)$. m-independence is similar in many ways to forking independence (see [Ne7]), and is related to $\mathcal{M}$-rank much like forking independence to U-rank. The next lemma contains some of the properties of $\mathcal{M}$-rank and m-independence, holding in a small superstable theory.

LEMMA 1.2 ([Ne2, Ne3]):
(1) (symmetry) *$a \overset{\mathrm{m}}{\perp} b(A)$ implies $b \overset{\mathrm{m}}{\perp} a(A)$.*
(2) (transitivity) *$ab \overset{\mathrm{m}}{\perp} c(A)$ iff $a \overset{\mathrm{m}}{\perp} c(Ab)$ and $b \overset{\mathrm{m}}{\perp} c(A)$.*

(3) $\mathcal{M}(a/A) \leq \mathcal{M}(ab/A) \leq \mathcal{M}(a/Ab) \oplus \mathcal{M}(b/A)$.
(4) *If* $a \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.8em\lower0.3ex\hbox{$|$}} b(A)$, *then* $\mathcal{M}(a/Ab) + \mathcal{M}(b/A) \leq \mathcal{M}(ab/A)$.
(5) *If* $a \overset{m}{\smile} b(A)$, *then* $\mathcal{M}(ab/A) = \mathcal{M}(a/A) \oplus \mathcal{M}(b/A)$ *and* $\mathcal{M}(a/A) = \mathcal{M}(a/Ab)$.
(6) *If* $\mathcal{M}(a/A) < \infty$ *and* $a \smile b(A)$ *then* $\mathcal{M}(a/A) = \mathcal{M}(a/Ab)$ *implies* $a \overset{m}{\smile} b(A)$.
(7) (existence of non-forking extensions of the same $\mathcal{M}$-rank) *If* $B \supset A$ *is finite, then for every* a *there is an* $a' \equiv a(A)$ *with* $a' \overset{m}{\smile} B(A)$ *(hence with* $\mathcal{M}(a'/A) = \mathcal{M}(a'/B)$*).*

Most importantly we have

THEOREM 1.3 ([Ne4, Ne6]): *If* T *has few countable models, then for any* p, $\mathcal{M}(p)$ *is finite and* $\mathcal{M}(p) \leq U(p)$.

We can measure also closed subsets of $S(\mathrm{acl}(A))$. If $\mathcal{P}$ is such a subset then $\mathcal{M}(\mathcal{P})$ is defined as $\max\{\mathcal{M}(a/B)\colon A \subseteq B,\ a \smile B(A)$ and $\mathrm{Tr}_A(a/B) \subseteq \mathcal{P}\}$. $\mathcal{M}$-rank considerations lead to notions of meager forking and meager type [Ne4].

Suppose $\mathcal{P}$ is a closed subset of $S(\mathrm{acl}(A))$. We say that forking is meager on $\mathcal{P}$ if for every formula $\varphi(x)$ forking over A, $\mathrm{Tr}_A(\varphi) \cap \mathcal{P}$ is meager (equivalently: nowhere dense) in $\mathcal{P}$.

Suppose p is a regular stationary type. We say that a formula $\varphi(x)$ over A is a p-formula (over A) if the following conditions hold.

(a) For each $a \in \varphi(\mathfrak{C})$, $\mathrm{tp}(a/\mathrm{acl}(A))$ is either regular and non-orthogonal to p or hereditarily orthogonal to p.

(b) The set $P_\varphi = \{r(x) \in S(\mathrm{acl}(A))\colon w_p(r) > 0\}$ is closed and non-empty.

(c) p-weight 0 is definable on $\varphi(\mathfrak{C})$, that is, if $a \in \varphi(\mathfrak{C})$ and $w_p(a/Ac) = 0$, then for some formula $\psi(x, y)$ over $\mathrm{acl}(A)$, true of (a, c), whenever $\psi(a', c')$ holds then $w_p(a'/Ac') = 0$.

By [HS], when p is non-trivial, p-formulas exist over many finite sets A. It is quite easy to find them when T is small [Ne4, lemma 1.6]. If p is non-trivial and weakly minimal, then any weakly minimal $\varphi(x) \in p$ is a p-formula. p-formulas are nice since within them we can work with cl_p just like with acl in the weakly minimal case.

We say that p is meager if for some (equivalently: any) p-formula φ (over some A), forking is meager on P_φ. We say that a complete type q is meager if every stationarization of q is meager. By [Ne4], if p is meager, then p is locally modular and non-trivial. So when p is meager we define Pm_φ as the set of modular types in P_φ. By [Ne4, corollary 1.8], Pm_φ is closed and nowhere dense in P_φ and $P_\varphi \setminus Pm_\varphi$ is open in $S(\mathrm{acl}(A))$.

Meager types may be thought of as generalizations of properly weakly minimal non-trivial types. In fact, each properly weakly minimal non-trivial type is meager. In a 1-based theory, each regular type is either meager or non-orthogonal to a strongly regular type [Ne6, beginning of section 3]. I conjecture this is true for an arbitrary superstable theory, under the few models assumption.

The following property of meager types is a generalized Saffe's condition.

THEOREM 1.4 ([Ne6]): *Assume T has few countable models, A is finite and $p \in S(A)$ is meager. Then exactly one of the following conditions holds.*

(1) *For some $a_1, \ldots, a_n$ realizing p, every $r \in \mathrm{Tr}_A(p)$ is realized in $\mathrm{cl}_p(Aa_i)$ for some i.*
(2) *p is isolated.*

This theorem says that a non-isolated meager type is small in some respect. We say that a meager $p \in S(A)$ is small if it satisfies (1) in the above theorem.

COROLLARY 1.5 ([Ne6]): *Assume T has few countable models, A is finite, $p \in S(A)$ is regular and forking is meager on $\mathrm{Tr}_A(p)$. Then*

(1) *p is meager and isolated, and*
(2) $\mathcal{M}(p) = 1 + \max\{\mathcal{M}(q)\colon q \in S_{p,nf}(AB)\colon B$ *is finite and q is small*$\}$.

(1) in the above corollary implies that the formula isolating p there is an r-formula for any stationarization r of p. $S_{p,nf}(AB)$ denotes the set of non-forking extensions of p in $S(AB)$.

Suppose p is meager. Since p is locally modular, there is a definable regular group G non-orthogonal to p. Hence given a regular definable abelian group G we say that G is meager if any generic type of G is meager. The notions of meager forking, meager type and meager group are defined in an arbitrary superstable theory.

In this paper we are concerned with meager groups. Now we introduce some notation (see [Ne4, Ne6]). Suppose G is a regular 0-definable abelian group with locally modular generic types and p^* is the generic type of G^0. Let $\mathcal{G}$ be the set of generic types of G and $\mathcal{G}m$ be the set of modular types in $\mathcal{G}$. Gm denotes the $\bigwedge$-definable over $\emptyset$ subgroup of G generated by $\mathcal{G}m$. For $p, q \in \mathcal{G}$, $p+q = \mathrm{stp}(a+b)$ for any independent realizations a, b of p, q respectively. For any A, $S_{gen}(A) = \{\mathrm{tp}(a/A)\colon a \in G$ is generic over $A\}$. Notice that G is a p^*-formula. In [Ne4] we prove that $\mathcal{G}m$ is closed and $\mathcal{G} \setminus \mathcal{G}m$ is open in $S(\mathrm{acl}(\emptyset))$. Moreover (under the few models assumption), G is meager iff $\mathcal{G}m$ is nowhere dense. We define $\mathcal{M}(G)$ as $\mathcal{M}(\mathcal{G})$.

By [Hr], cl_{p^*} on G^0 is essentially a vector space dependence over a division ring F_G of definable pseudo-endomorphisms of G. When G is meager, then every element of F_G is $\mathrm{acl}(\emptyset)$-definable (or rather: is equivalent to an $\mathrm{acl}(\emptyset)$-definable one). In this case F_G is countable, and even (by smallness of T) is a locally finite field (hence has non-zero characteristic) (see [Lo, Ne4]). In the context of G we can restate Theorem 1.4 and Corollary 1.5 as follows.

Suppose $p \in S_{gen}(A)$. Then p is meager, so Theorem 1.4 applies. If (2) holds, then p is isolated. Otherwise (1) holds; however, in [Ne4] it is proved that within G, $\mathrm{cl}_p(Aa_i) \cap \mathcal{G}$ is a finite union of cosets of $\mathcal{G}m$. So we have the following theorem.

THEOREM 1.6 ([Ne6]): *Assume T has few countable models, A is finite, G is a 0-definable meager group and $p \in S_{gen}(A)$. Then either p is isolated or* $\mathrm{Tr}(p) \subseteq \bigcup_i r_i + \mathcal{G}m$ *for some finitely many $r_1, \ldots, r_n \in \mathrm{Tr}(p)$.*

Since $\mathcal{M}(G)$ equals $\mathcal{M}(p)$ for any isolated $p \in S_{gen}(\emptyset)$, and for a locally modular abelian G, G is meager iff $\mathcal{G}m$ is nowhere dense, we have the following corollary.

COROLLARY 1.7 ([Ne6]): *Assume T has few countable models and G is a 0-definable locally modular abelian group. Then $\mathcal{M}(G) = \mathcal{M}(Gm) + 1$ when G is meager, and $\mathcal{M}(G) = \mathcal{M}(Gm)$ otherwise.*

Suppose there is a non-generic 0-definable subgroup H of G such that for a realizing p^*, $a + H$ contains a realization of any type in $\mathcal{G}m$. Let $G' = G/H$. Clearly G' is non-orthogonal to G. In this situation Corollary 1.7 yields that $\mathcal{M}(G') = 0$, that is, G' is strongly regular (in the case, when G is not meager) and G' is meager with $\mathcal{M}(G') = 1$ (when G is meager). Due to the form of definable sets in 1-based groups [HP], when T is 1-based, such an H exists. So we get the following corollary.

COROLLARY 1.8 ([Ne6]): *Assume T is 1-based, with few countable models, and G is a 0-definable locally modular abelian group. Then there is a non-generic 0-definable subgroup H of G such that:*

(1) *if G is meager then G/H is meager and $\mathcal{M}(G/H) = 1$,*
(2) *if G is not meager then G/H is strongly regular, hence $\mathcal{M}(G/H) = 0$.*

There is also a group-free version of this corollary [Ne6, 3.2], but we shall not use it here. Corollary 1.8 justifies our interest in meager groups of $\mathcal{M}$-rank 1. Suppose G is a meager group of $\mathcal{M}$-rank 1. In this paper we try to classify countable models of $\mathrm{Th}(G)$; in other words, to classify sets of the form $G(M)$, where M is a countable model of T. Our classification relies on the geometrical properties of cl_{p^*}, and is relative to $\mathrm{cl}_{p^*}(\emptyset) \cap G(M)$. Notice that

$\mathrm{cl}_{p^*}(\emptyset) \cap G$ is a countable union of 0-definable sets. To simplify the proofs, we add to $\mathrm{cl}_{p^*}(\emptyset) \cap G$ the set $\mathrm{acl}(\emptyset)$. So let Φ be a countable disjunction of formulas defining $\mathrm{acl}(\emptyset) \cup (G \cap \mathrm{cl}_{p^*}(\emptyset))$, let M^* be a countable model of T and let $Q = \Phi(M^*)$. We try to count the groups of the form $G(M)$, where M is countable and $Q = \Phi(M)$.

In general we succeed only partially: we manage to count sets of the form $\mathcal{G}^M$, where M is countable and $\Phi(M) = Q$. Here $\mathcal{G}^M$ is the set of types $r \in \mathcal{G}$ realized in M. In case when the ring of pseudo-endomorphisms F_G is particularly simple (e.g. is just a prime field), this yields a classification of the sets $G(M)$, and relative Vaught's conjecture over Q.

Q may be regarded as a model of a many-sorted theory $T\lceil\Phi$ (cf. [Pi]). When G has U-rank ω, then $T\lceil\Phi$ has finite U-rank and Vaught's conjecture is proved for $T\lceil\Phi$ [Bu2]. Hence in this case we get Vaught's conjecture for $\mathrm{Th}(G)$.

The proofs essentially generalize [Bu1] (Buechler considered the weakly minimal case there) and [Pi] (Pillay considered a meager group G with the property that $T\lceil\Phi$ is ω-stable; in this case necessarily $\mathcal{M}(G) = 1$). Pillay's proof relied heavily on the assumption that $T\lceil\Phi$ is ω-stable. Here we use instead the properties of $\mathcal{M}$-rank mentioned above. In [Pi], at the final stage, $G(M)$ is proved to be atomic over a countable skeleton (built up from some Morley sequences). Here we show that $G(M)$ is Q-atomic over some Morley sequence. In virtue of Lemma 1.5, this is enough.

2. A basis lemma

From now on in this paper we assume T has few countable models. In this section we generalize the following result of Steven Buechler [Bu1, lemma 5.2]:

(*) If M is a countable model of T, then there is a finite set $A \subseteq M$ of elements realizing properly weakly minimal types, such that for every $a \in M$ with $\mathcal{M}(a/\emptyset) = 0$ and $r = \mathrm{stp}(a)$ properly weakly minimal, $r|A$ is modular (if r is not modular, this implies r is realized in $\mathrm{acl}(A)$).

We deal with the following situation. $\Phi(x)$ is a countable disjunction of formulas over $\emptyset$, $Q = \Phi(M^*)$ for some countable model M^* of T and $\mathrm{acl}(\emptyset) \subseteq Q$. We are interested in countable models $M \in K_Q$. We shall generalize (*) for such M by replacing properly weakly minimal types by some locally modular types, which are orthogonal to Φ. (*) says that within a model M there is a "basis" (the set A) for certain small properly weakly minimal types realized in M. Here "small" means "having finite trace". When we work in K_Q, it is the size of the Aut(Q)-orbits (pseudo-types) that provides the dividing line between "small" and "large" types. So we are looking for a finite "basis" for all small Aut(Q)-orbits of some locally modular types.

We shall need some additional analysis of traces of non-isolated types in a p-formula. Suppose p is a stationary meager type and $\varphi(x)$ is a p-formula over $\emptyset$. For any finite set B let

$$d(B) = \bigcup\{\mathrm{Tr}(a/B)\colon \mathrm{stp}(a) \in P_\varphi, a \perp B \text{ and } \mathrm{tp}(a/B) \text{ is non-isolated}\}$$

and

$$\mathrm{CL}(B) = \{r \in P_\varphi\colon r|B \text{ is modular}\}.$$

So $r \in \mathrm{CL}(B)$ iff $r \in P_\varphi$ and r is modular or r is realized in $\mathrm{cl}_p(B)$. Also, by lemma 2.13 from [Ne4], $\mathrm{CL}(B)$ is closed nowhere dense in P_φ, $Pm_\varphi = \mathrm{CL}(\emptyset)$ is open in $\mathrm{CL}(B)$ and for some $a_0, \ldots, a_{k-1}$ with $\mathrm{stp}(a_i) \in \mathrm{CL}(B)$ we have $\mathrm{CL}(B) = \bigcup_{i<k} \mathrm{CL}(a_i)$.

Lemma 2.1:

(1) *$d(B)$ is closed and nowhere dense.*

(2) $\mathrm{CL}(B \cup d(B)(\mathfrak{C})) = d(B)$. *In particular,* $\mathrm{CL}(B) \subseteq d(B)$.

Proof: (1) The proof is similar as in [Ne4, corollary 1.8], and uses smallness of T.

(2) Suppose $r \in \mathrm{CL}(B \cup d(B)(\mathfrak{C}))$. Wlog r is not modular. So there are $a_1, \ldots, a_n$ realizing types in $d(B)$ such that r is realized in $\mathrm{cl}_p(Ba_1, \ldots, a_n)$, i.e. $r \in \mathrm{CL}(Ba_1 \ldots a_n)$. Wlog $a_i \perp B$. Let $q_i = \mathrm{tp}(a_i/B)$. So q_i is non-isolated. Let $X = \bigcup\{\mathrm{CL}(Ba'_1 \ldots a'_n)\colon a'_i \text{ realizes } q_i\}$. Since q_i is non-isolated, by Theorem 1.4,

$$\bigcup\{\mathrm{CL}(a'_i)\colon a'_i \text{ realizes } q_i\} = \bigcup_{j<k_i} \mathrm{CL}(b^i_j)$$

for some $b^i_0, \ldots, b^i_{k_i-1}$ realizing q_i. Hence

$$X = \bigcup_{j_1<k_1} \bigcup_{j_2<k_2} \cdots \bigcup_{j_n<k_n} \mathrm{CL}(Bb^1_{j_1} b^2_{j_2} \cdots b^n_{j_n}).$$

By [Ne4, lemma 2.13], $\mathrm{CL}(Bb^1_{j_1} \cdots b^n_{j_n})$ is closed nowhere dense, hence also X is closed, nowhere dense and $\mathrm{Aut}(\mathfrak{C}/B)$-invariant. So clearly $X \subseteq d(B)$. Since $r \in \mathrm{CL}(Ba_1 \cdots a_n) \subseteq X$, we are done.

It may happen that $\mathrm{CL}(B) \neq d(B)$. The next proposition is similar to [Ne4, lemma 2.13], which deals with $\mathrm{CL}(B)$.

PROPOSITION 2.2: *There are finitely many elements $a_1, \dots, a_k$ realizing types in $d(B)$ such that $d(B) = \bigcup_i \mathrm{CL}(a_i)$.*

Proof: The proof consists in reducing the proposition to the case when φ defines a group, and in this case the proposition is easy.

By [Ne6, corollary 1.3] there is a meager group G definable over $\mathrm{cl}_p(\emptyset)$, which is non-orthogonal to p. Since naming $\mathrm{cl}_p(\emptyset)$ does not affect CL, we can assume that G is 0-definable. Increasing B, we can assume that every $r \in P_\varphi$ is not almost orthogonal (over B) to some $r' \in \mathcal{G}$, and vice versa.

Let $d'(B), \mathrm{CL}'(B)$ be defined as d and CL, but with respect to G and $\mathcal{G}$ instead of φ, P_φ. We have that

$$d'(B) = \bigcup \{\mathrm{Tr}(q) \colon q \in S_{gen}(B) \text{ and } q \text{ is non-isolated}\}.$$

For $q \in S_{gen}(B)$ let $X(q)$ be the union of $\mathcal{G}m$-cosets meeting $\mathrm{Tr}(q)$. By Theorem 1.6, if $q \in S_{gen}(B)$ is non-isolated then $X(q)/\mathcal{G}m$ is finite, hence $X(q)$ is nowhere dense. Also, $X(q)$ is B-invariant, hence if q is non-isolated then $X(q) \subseteq d'(B)$. This shows that $d'(B)$ is a union of some number of $\mathcal{G}m$-cosets, and, by Lemma 2.1, $d'(B)$ is closed and nowhere dense. We see that for $r \in \mathcal{G}$, $r \in d'(B)$ iff the $\mathrm{Aut}(\mathfrak{C}/B)$-orbit of $r/\mathcal{G}m$ is finite. It follows that for $r, r' \in d'(B)$, $r + r' \in d'(B)$ and $(-r) \in d'(B)$. So $d'(B)$ is the set $\mathcal{G}'$ of generic types of some generic subgroup G' with $Gm \subseteq G' \subseteq G$. By Corollary 1.7, $\mathcal{M}(Gm) + 1 = \mathcal{M}(G)$. Since $\mathcal{G}'$ is nowhere dense in $\mathcal{G}$, this implies that Gm has finite index in G' (see [Ne6]); in other words, there are finitely many generic $b_0, \dots, b_k \in G'$ with $d'(B) = \mathcal{G}' = \bigcup_i (\mathcal{G}m + \mathrm{stp}(b_i)) = \bigcup_i \mathrm{CL}'(b_i)$. Choose a_i realizing a type in p_φ with $a_i \not\perp b_i$. By Lemma 2.1 (applied to $\varphi'(x) = \varphi(x) \vee G(x)$) we get that $d(B) = \bigcup_i \mathrm{CL}(a_i)$.

Suppose $p \in S(\emptyset)$ is locally modular. Then there are three cases:

(1) p is not meager,

(2) p is meager and non-isolated,

(3) p is meager and isolated.

The next lemma deals with $\mathrm{Tr}(p)$ in the first two cases.

LEMMA 2.3: *Assume $p \in S(\emptyset)$ is locally modular.*

(1) *If p is not meager, then there is a finite set B of independent realizations of p such that every $r \in \mathrm{Tr}(p)$ has a forking extension over B.*

(2) *If p is trivial or (meager and non-isolated), then there is a finite set B as above , such that every $r \in \mathrm{Tr}(p)$ has a forking extension over some $b \in B$.*

Proof: (1) If p is not meager, by Corollary 1.5 forking is not meager on $\mathrm{Tr}(p)$. Revealing the definition we get the required B.

(2) If p is trivial, we proceed as in [Ne4, lemma 1.6(1)] (see also the proof of theorem 2.6 in [Ne6]). If p is meager and non-isolated, the conclusion follows by Theorem 1.4.

Now suppose $p \in S(\emptyset)$ is locally modular orthogonal to Φ. Then for any $r \in \mathrm{Tr}(p)$, $r \vdash r|Q$. So $\mathrm{Aut}(Q)$ acts on $\mathrm{Tr}(p)$, and we are interested in the orbits of this action (pseudo-types). Notice that the topology on $\mathrm{Tr}(p)$ is metrizable, say, by the metric ρ. The next lemma deals with the case when p is meager and isolated. In the proof we will need the following claim.

CLAIM 2.4: $\mathrm{Aut}(Q)$, *regarded as a family of mappings on* $\mathrm{Tr}(p)$, *is uniformly continuous, that is, for every* $\epsilon > 0$ *there is* $\epsilon' > 0$ *such that for every* $f \in \mathrm{Aut}(Q)$ *and all* $x, y \in \mathrm{Tr}(p)$, *if* $\rho(x,y) < \epsilon'$ *then* $\rho(f(x), f(y)) < \epsilon$.

Proof: Notice that for every formula $\delta(x,y)$ the set $D = \{r|\delta\colon r \in \mathrm{Tr}(p)\}$ is finite. Also, for any $r \in \mathrm{Tr}(p)$ the set $D_{r,\delta} = \{r' \in \mathrm{Tr}(p)\colon r|\delta = r'|\delta\}$ is clopen in $\mathrm{Tr}(p)$. Moreover, sets of the form $D_{r,\delta}$ are a basis of the topology on $\mathrm{Tr}(p)$. Clearly, if $f \in \mathrm{Aut}(Q)$ and $r|\delta = r'|\delta$ then $f(r)|\delta = f(r')|\delta$. This proves the claim.

LEMMA 2.5: *Assume* $p \in S(\emptyset)$ *is meager, isolated and orthogonal to* Φ, $r^* \in \mathrm{Tr}(p)$, X *is the* $\mathrm{Aut}(Q)$*-orbit of* r^*. *Then either* X *is open in* $\mathrm{Tr}(p)$ *or there are finitely many types* $r_i \in X$ *such that every* $r \in X$ *is not almost orthogonal to some* r_i. *In the latter case* X *is nowhere dense.*

Proof: Wlog all stationarizations of p are non-orthogonal. First we prove that

(a) if X is meager, then there are finitely many types $r_i \in X$ such that every $r \in X$ is not almost orthogonal to some r_i.

Suppose (a) fails. We shall construct many models of T. Let a realize a type in X. We can choose isolated types $p_n \in S(a), n < \omega$, such that for every n, p_n is a non-forking extension of p, $X_n = \mathrm{Tr}(p_n) \cap X$ is non-empty and $\{\mathrm{stp}(a)\} = \bigcap_k \bigcup_{n>k} \mathrm{Tr}(p_n)$.

Indeed, if for some open neighbourhood U of $\mathrm{stp}(a)$ we have that $U \cap X$ is disjoint from $\mathrm{Tr}(p')$ for any isolated $p' \in S_{p,nf}(a)$, then we get that $U \cap X \subseteq d(a)$, hence by Proposition 2.2 there are finitely many types $r_i \in U \cap X$ such that every $r \in U \cap X$ is not almost orthogonal to some r_i. Since X may be covered by finitely many $\mathrm{Aut}(Q)$-translates of U, we would get (a).

We shall find an almost orthogonal subfamily $\{p_{n_i}, i < \omega\}$. We define recursively an increasing sequence of natural numbers $n_i, i < \omega$ so that for every $i < \omega$ the following holds:

(b) If $a_0, \dots, a_i$ realize $p_{n_0}, \dots, p_{n_i}$ then $\mathrm{stp}(a_i) \notin d(aa_0, \dots, a_{i-1})$.

Suppose we have defined $n_0, \dots, n_j$ so that for all $i \leq j$, (b) holds. This implies that for every $i \leq j$, the set of types $Z_i = \{\mathrm{tp}(a'_0 \dots a'_i/a)$: a'_t realizes p_{n_t} for $t \leq i\}$ is finite.

Indeed, Z_0 is finite trivially. Suppose $i < j$ and Z_i is finite, say $Z_i = \{\mathrm{tp}(b_t/a)$: $t < k_i\}$ for some k_i. By (b), $\mathrm{Tr}(p_{n_{i+1}}) \cap d(ab_t) = \emptyset$ for every t. It follows that $p_{n_{i+1}}$ has only finitely many non-forking extensions over ab_t, and all of them are isolated, hence there are finitely many of them. Thus Z_{i+1} is finite.

Let $Z_j = \{\mathrm{tp}(b_t/a)$: $t < k_j\}$. We have that $d(ab_t)$ meets $\mathrm{Tr}(p_n)$ for at most finitely many n. Indeed, by Proposition 2.2, $d(ab_t)$ is a finite union of sets of the form $\mathrm{CL}(c)$, where $\mathrm{stp}(c) \in d(ab_t)$, so if $d(ab_t)$ meets $\mathrm{Tr}(p_n)$ for arbitrarily large n, then for some c realizing some p_n, $\mathrm{CL}(c)$ meets $\mathrm{Tr}(p_n)$ for arbitrarily large n. Since $\{\mathrm{stp}(a)\} = \bigcap_k \bigcup_{n>k} \mathrm{Tr}(p_n)$ and $\mathrm{CL}(c)$ is closed, we get $\mathrm{stp}(a) \in \mathrm{CL}(c)$, hence $a \not\perp^{m} c$. But p_n is isolated, so $c \perp^{m} a$, a contradiction.

So we can choose $n_{j+1} > n_j$ such that for every t, $d(ab_t) \cap \mathrm{Tr}(p_{n_{j+1}}) = \emptyset$. We see that (b) holds with this choice of n_{j+1} for $i = j+1$.

Clearly, (b) implies that $\{p_{n_i}, i < \omega\}$ is an almost orthogonal family of types. Choose a_i realizing a type in X_{n_i}. By the omitting types theorem, for any $I \subseteq \omega$ we find a model $M_I \in K_Q$ containing a and $a_i, i \in I$, and omitting every type in $X_{n_i}, i \notin I$. Indeed, since X_{n_i} is meager, it is covered by a countable union of closed nowhere dense subsets of $\mathrm{Tr}(p_{n_i})$. Each such subset of $\mathrm{Tr}(p_{n_i})$ is of the form $\mathrm{Tr}(q)$ for some non-isolated (possibly incomplete) type q over $\mathrm{acl}(\emptyset)$ and since $\mathrm{Tr}(q) \subseteq \mathrm{Tr}(p_{n_i})$ and $p_{n_i} \in S(a)$ is isolated, q has no forking extension over a (otherwise each type in $\mathrm{Tr}(p_{n_i})$ would have a forking extension over a, contradicting the meager forking assumption). Since $\{p_{n_i}, i < \omega\}$ is almost orthogonal, q has no forking extension over $\{a\} \cup \{a_{n_i}: i \in I\} \cup Q$, hence is non-isolated over $\{a\} \cup \{a_{n_i}: i \in I\} \cup Q$. We can omit the countably many types q by a model in K_Q.

Hence for $I \neq I' \subseteq \omega$, $M_I, M_{I'}$ are non-isomorphic (over a). So we get $2^{\aleph_0}$-many countable models. This contradiction proves (a).

Now for every $r \in \mathrm{Tr}(p)$ the set of $r' \in \mathrm{Tr}(p)$ not almost orthogonal to r is closed and nowhere dense ([Ne4, lemmas 1.8, 2.13]), so by (a) if X is meager then X is nowhere dense. So to prove the lemma it is enough to show that

(c) X is either open or nowhere dense.

Suppose not. Let $X' = \mathrm{int}(\mathrm{cl}(X))$. Notice that since X is an $\mathrm{Aut}(Q)$-orbit, for any $r \in X'$, the $\mathrm{Aut}(Q)$-orbit of r is contained in X' and dense in X' (this follows from Claim 2.4).

X, being Borel, has the Baire property, so is co-meager in some open set $U \subseteq S(\mathrm{acl}(\emptyset))$. Clearly $U \subseteq \mathrm{cl}(X)$. It follows that $X \subseteq X'$ and X is co-meager in X'. Since X is not open, $X' \setminus X \neq \emptyset$. Let X^* be any orbit contained in $X' \setminus X$. So X^* is meager and dense in X', hence is not nowhere dense. This contradicts (a) (modulo the remark before (c)).

The following is a generalization of $(*)$.

THEOREM 2.6: *Assume $\Phi(x)$ is a countable disjunction of formulas over $\emptyset$ such that* $\mathrm{acl}(\emptyset) \subseteq \Phi(\mathfrak{C})$, *$M$ is a countable model of T, $Q = \Phi(M)$, and let $(**)$ be the following condition:*

$(**)$ *$r \in S(\mathrm{acl}(\emptyset))$ is locally modular orthogonal to Φ and the* $\mathrm{Aut}(Q)$*-orbit of r is meager.*

Then there is a finite independent set $B_M \subseteq M$ of elements realizing types r satisfying $(**)$ and such that, for any r satisfying $(**)$, if r is realized in M then r has a forking extension over B_M.

Proof: It is easy to see that it suffices to find a finite independent set B of elements realizing types with $(**)$ such that

(a) every type r with $(**)$ has a forking extension over B.

Indeed, suppose we have such a B. Since $w(B)$ is finite we get that there is no almost orthogonal family $\{r_n, n < \omega\}$ of types with $(**)$. It follows that within any M as in the theorem we can find the required set B_M.

By Lemmas 2.3 and 2.5, if X is the $\mathrm{Aut}(Q)$-orbit of a type r satisfying $(**)$, then there is a finite set B_X of independent realizations of types in $\mathrm{cl}(X)$ such that any $r' \in \mathrm{cl}(X)$ has a forking extension over B_X.

By [Ne3], there are countably many sets of the form $\mathrm{cl}(X)$. Indeed, $\mathrm{cl}(X)$ is $\mathrm{Aut}(Q)$-invariant and closed. Also, all the $\mathrm{Aut}(Q)$-orbits contained in $\mathrm{cl}(X)$ are dense in $\mathrm{cl}(X)$ (by Claim 2.4), so $\mathrm{cl}(X)$ contains a unique co-meager $\mathrm{Aut}(Q)$-orbit (by [Ne3], this is a variant of Lemma 1.1(3)). Clearly this $\mathrm{Aut}(Q)$-orbit is τ-stable, and by [Ne3] there are countably many τ-stable good pseudo-types.

Choose countably many $\mathrm{Aut}(Q)$-orbits $X_n, n < \omega$, of types r with $(**)$, such that for every orbit Y of a type with $(**)$, $\mathrm{cl}(Y) = \mathrm{cl}(X_n)$ for some n. Let $(\dagger)$ be the following statement:

(†) For every finite set B of independent realizations of types with $(**)$, for infinitely many $n < \omega$, no $r \in \mathrm{cl}(X_i)$ has a forking extension over B.

CLAIM 2.7: *$\neg$(†) implies* (a).

Proof: Suppose (†) fails for some finite set B. Extending B a little we can assume that for any $n < \omega$, some $r_n \in \mathrm{cl}(X_n)$ has a forking extension over B. Let X'_n be the $\mathrm{Aut}(Q)$-orbit of r_n. By Claim 2.4, X'_n is dense in X_n. It follows that for every $r \in X'_n$ there is some $B' = f[B]$ (for some $f \in \mathrm{Aut}(Q)$) such that r has a forking extension over B'. Suppose $B = \{b_0, \ldots, b_k\}$ and let O_i be the $\mathrm{Aut}(Q)$-orbit of $\mathrm{stp}(b_i), i \leq k$. Choose a finite set C_0 of independent realizations of types in $\bigcup_i O_i$ such that every $s \in \bigcup_i O_i$ has a forking extension over C_0. Let $C_0, \ldots, C_k$ be a Morley sequence in $\mathrm{stp}(C_0)$ and let $C = \bigcup_i C_i$. To prove the claim it is enough to show that

(b) every $r \in \mathrm{cl}(X_n)$ has a forking extension over C.

First assume $r \in X'_n$. Choose $f \in \mathrm{Aut}(Q)$ and $B' = f[B]$, $B' = \{b'_0, \ldots, b'_k\}$ such that r has a forking extension over B'. Suppose wlog that r is non-orthogonal to $\mathrm{stp}(b'_0), \ldots, \mathrm{stp}(b'_i)$ and orthogonal to $\mathrm{stp}(b'_{i+1}), \ldots, \mathrm{stp}(b'_k)$. Choose $b''_i \overset{s}{\equiv} b'_i$ with $b''_i \not\perp C_i$ and such that $\{C_i b''_i \colon i \leq k\}$ is independent. Let $B'' = \{b''_0, \ldots, b''_k\}$. Clearly, for some a realizing r, $a \not\perp B''$. If $a \perp C_0 \cdots C_i$ then $a \perp C_0 \cdots C_i b''_{i+1} \cdots b''_k$. Since $b''_0, \ldots, b''_i$ depend on $C_0 \cdots C_i$, also $a \perp B'' C_o \cdots C_i$, hence $a \perp B''$, a contradiction. So $a \not\perp C$ showing that r has a forking extension over C.

Now let Y_n be the set of $r \in \mathrm{cl}(X_n)$ with a forking extension over C. Since X'_n is dense in $\mathrm{cl}(X_n)$, also Y_n is dense in $\mathrm{cl}(X_n)$. Choose $p \in S(\emptyset)$ with $X_n \subseteq \mathrm{Tr}(p)$. We need the following subclaim.

SUBCLAIM 2.8: *The set Z of types $r \in \mathrm{Tr}(p)$ with a forking extension over C is closed.*

Proof: We can assume that all types in $\mathrm{Tr}(p)$ are non-orthogonal (anyway the non-orthogonality classes on $\mathrm{Tr}(p)$ are clopen in $\mathrm{Tr}(p)$). The case when p is modular is easy. So assume p is non-modular.

If p is isolated and meager, we are done by [Ne4, lemma 2.13] (since $Z = \mathrm{CL}(C)$).

Suppose p is not isolated or not meager. Let $Z' = \mathrm{cl}(Z)$. By smallness, Z has non-empty interior in Z', that is, for some (relatively) open $U \subseteq Z'$ we have $U \subseteq Z$.

By Lemma 2.3 there are finitely many almost orthogonal $r_i \in Z$ such that for every $r' \in Z'$ we have

(c) $r' \overset{a}{\not\perp} \otimes_i r_i$.

To find the r_i, first find finitely many almost orthogonal $r_i \in U$ such that for any $r' \in U$, (c) holds. Then add finitely many almost orthogonal $r_i \in Z$ to ensure that (c) holds for every $r' \in Z'$.

By (c), $Z' = Z$ and the subclaim is proved.

By the subclaim we see that Y_n is closed, hence $Y_n = \mathrm{cl}(X_n)$, proving (b) and the claim.

By Claim 2.7, in the further proof of Theorem 2.6 we may assume that (†) holds. In this case we will reach a contradiction constructing many models of T.

We find recursively an increasing sequence $n_i, i < \omega$ and finite sets B_i such that the following hold:

(d) B_i is an independent set of realizations of types in $\mathrm{cl}(X_{n_i})$ and every type $r \in \mathrm{cl}(X_{n_i})$ has a forking extension over $\bigcup_{j \leq i} B_j$.

(e) $\bigcup_i B_i$ is independent.

(f) No $r \in \mathrm{cl}(X_{n_i})$ has a forking extension over $\bigcup_{j<i} B_j$.

It is clear how to satisfy (d), (e). If at some point no n_i can be chosen to satisfy (f), this would mean that for some co-finite set $I \subset \omega$, for every $m \in I$, some $r \in \mathrm{cl}(X_n)$ has a forking extension over $B = \bigcup_{j<i} B_j$, contradicting (†).

Choose a_i realizing a type in X_{n_i}. By (d), (e), (f), the set $\{a_i, i \in I\}$ is independent and no type in $\mathrm{cl}(X_{n_i})$, $i \notin I$, has a forking extension over it. We see that for any $I \subseteq \omega$ there is a model $M_I \in K_Q$ containing $a_i, i \in I$, and omitting every type in $X_{n_i}, i \notin I$ (as in the proof of Lemma 2.5). So we get many countable models of T.

3. A ground level lemma

In this section we adapt to our context theorem 4.1 from [Bu1], which we state here in the following form:

($*$) Suppose Aa is a finite independent set of elements realizing over $\emptyset$ properly weakly minimal types. If $\mathcal{M}(a/A) = 0$ and $\mathcal{M}(a/\emptyset) \neq 0$, then either $\mathrm{stp}(a/A)$ is modular, or else, for some b with $\mathrm{tp}(b)$ weakly minimal and $\mathcal{M}(b/\emptyset) = 0$, a and b are interalgebraic over A.

Existence of an a as in ($*$) is a major obstacle in [Bu1] in showing that M is prime over some independent set A. ($*$) says how to deal with this case: this obstacle is pulled down to the ground level, i.e. a is replaced by some b with $\mathcal{M}(b/\emptyset) = 0$, and then b is dealt with using the basis lemma from the previous section.

In this section we fix the following set-up. Φ is a countable disjunction of some formulas over $\emptyset$, $\mathrm{acl}(\emptyset) \subseteq \Phi(\mathfrak{C})$ and $Q = \Phi(M^*)$ for some countable model M^*. If $r \in S(\mathrm{acl}(\emptyset))$ and C is finite, we call the $\mathrm{Aut}(Q/C)$-orbit of r an $\mathrm{Aut}(Q)$-orbit of r over C. We say that r is Q-finite over C if this orbit is finite. We fix a meager stationary type $p \in S(\emptyset)$, which is orthogonal to Φ. We are interested in p-formulas φ over $\mathrm{acl}(\emptyset)$ satisfying $\mathcal{M}(P_\varphi) = 1$. So in this section we say that a is regular if $\mathrm{stp}(a)$ is regular non-orthogonal to p and a realizes some p-formula $\varphi(x)$ over $\mathrm{acl}(\emptyset)$ with $\mathcal{M}(P_\varphi) = 1$. We say that a is Q-finite over C if $\mathrm{stp}(a)$ is Q-finite over C. We can restate Lemma 2.5 in this set-up as follows.

LEMMA 3.1: *Assume a is regular.*

1. *Either* $\mathrm{tp}(a)$ *is isolated (and then* $\mathcal{M}(a/\emptyset) = 1$*) or* $\mathcal{M}(a/\emptyset) = 0$.
2. *The* $\mathrm{Aut}(Q)$*-orbit of* $\mathrm{stp}(a)$ *is either open or finite.*
3. $\mathrm{tp}(a/Q)$ *is Q-isolated iff a is not Q-finite iff the* $\mathrm{Aut}(Q)$*-orbit of* $\mathrm{stp}(a)$ *is open.*

Proof: Choose a p-formula $\varphi(x)$ over $\mathrm{acl}(\emptyset)$, true of a. Wlog φ is a p-formula over $\emptyset$. So we have that $\mathrm{Tr}(a/\emptyset) \subseteq P_\varphi$, $\mathcal{M}(P_\varphi) = 1$ and $P'_\varphi = P_\varphi \setminus Pm_\varphi$ is open in $S(\mathrm{acl}(\emptyset))$. Also Pm_φ is closed and nowhere dense in P_φ. So $\mathcal{M}(a/\emptyset) \leq 1$.

(1) If $\mathrm{tp}(a)$ is isolated then $\mathrm{Tr}(a/\emptyset)$ is clopen in P_φ. Forking is meager on P_φ, hence $\mathcal{M}(a/\emptyset) > 0$, i.e. $\mathcal{M}(a/\emptyset) = 1$. Suppose $\mathrm{tp}(a)$ is non-isolated and $\mathrm{Tr}(a/\emptyset)$ is infinite. By smallness choose a type $q \in S(\emptyset)$ with $\mathrm{Tr}(q)$ clopen in P'_φ. It follows that q is isolated, hence $\mathcal{M}(q) = 1$. Let b realize q. It is not hard to find over ab a non-forking extension q' of q ("translating" $\mathrm{Tr}(a/\emptyset)$ into $\mathrm{Tr}(q)$) such that $\mathrm{Tr}(q')$ is infinite and nowhere dense in $\mathrm{Tr}(q)$. Hence $\mathcal{M}(q) > \mathcal{M}(q') \geq 1$, a contradiction.

(2) If $\mathrm{tp}(a)$ is non-isolated, then, by (1) $\mathcal{M}(a/\emptyset) = 0$ and we are done.

If $\mathrm{tp}(a)$ is isolated then we can apply Lemma 2.5. Let X be the $\mathrm{Aut}(Q)$-orbit of $\mathrm{stp}(a)$. If X is not open then there are finitely many types $r_i \in X$ such that every type $r' \in X$ is not almost orthogonal to some r_i. But for each i, the set X_i of types in P_φ not almost orthogonal to r_i is finite (since it is closed and $\mathcal{M}(P_\varphi) = 1$). So X is finite.

(3) Since p is orthogonal to Φ, for any $r \in P_\varphi$, $r \vdash r|Q$. So the $\mathrm{Aut}(Q)$-orbit of r is homeomorphic to the $\mathrm{Aut}(Q)$-orbit of $r|Q$. If a is Q-finite then the $\mathrm{Aut}(Q)$-orbit of $\mathrm{stp}(a)$ is a finite subset of P_φ, hence is nowhere dense in P_φ, i.e. $\mathrm{tp}(a/Q)$ is not Q-isolated. If a is not Q-finite, then the $\mathrm{Aut}(Q)$-orbit of $\mathrm{stp}(a)$ is open, so $\mathrm{tp}(a/Q)$ is Q-isolated.

The following is a generalization of $(*)$. It is also a generalization of lemma 2.1 of [Pi]: the conclusion is roughly the same, but our assumptions are weaker. Pillay assumes additionally that each forking extension of $\mathrm{stp}(a)$ (for any regular a) has Morley rank, and considers the case $Q = \emptyset$.

THEOREM 3.2: *Suppose Aa is a finite independent set of regular elements, a is Q-finite over A, but not Q-finite over $\emptyset$. Then either $\mathrm{stp}(a/A)$ is modular or, for some regular Q-finite b independent of A, $a \not\perp b(A)$.*

Proof: Suppose $\mathrm{stp}(a/A)$ is not modular. As in the case when $Q = \emptyset$ one can prove that if $b \in A$ and b is Q-finite over $A \setminus \{b\}$, then a is Q-finite over $A \setminus \{b\}$. So choose a minimal subset A' of A such that for every $a' \in A \setminus A'$, a' is Q-finite over A'. As in the case of types of finite multiplicity it follows that a is Q-finite over A'. So wlog $A = A'$.

Let $c \in A$ and $B = A \setminus \{c\}$. We can also assume that a is not Q-finite over B. Hence the $\mathrm{Aut}(Q)$-orbit of a over B is open. Let X be the $\mathrm{Aut}(Q)$-orbit of $\mathrm{stp}(c)$ over B. We see that X is open in $S(\mathrm{acl}(\emptyset))$. Expanding the signature by an element of $\mathrm{acl}(\emptyset)$ we can assume that the $\mathrm{Aut}(Q)$-orbit of $\mathrm{stp}(a)$ over Bc is $\{\mathrm{stp}(a)\}$. Clearly all types in X are non-orthogonal.

CLAIM 3.3: *If C is a finite B-independent set of realizations of types in X, then the set of types $r \in X$ with finite $\mathrm{Aut}(Q)$-orbit over BC has no accumulation point in X.*

Proof: The proof is similar, e.g., to that of [Ne7, lemma 2.1]. To simplify notation, absorb B into the signature. Let $Y = \{r \in X\colon$ the $\mathrm{Aut}(Q)$-orbit of r over C is finite$\}$ and $Z = \mathrm{cl}(Y) \cap X$. We want to show that every point of Z is isolated (in Z).

Z is $\mathrm{Aut}(Q)$-invariant over C, hence is a union of some orbits over C. None of these orbits is open (since Y is dense in Z), so by Lemma 3.1 each of them is finite and $Z = Y$. Also, the perfect core Z' of Z is empty. (Otherwise Z' has power continuum and contains an orbit over C, which is not meager in Z'. This orbit is neither finite nor open in $S(\mathrm{acl}(\emptyset))$, contradicting Lemma 3.1.) Hence Z is countable and points isolated in Z are dense in Z.

Suppose $r \in Z$ is an accumulation point of Z. Pick an isolated $r' \in Z$. Using local modularity and basic properties of meager types it is easy to "translate" a sequence of types from Z converging to r into a sequence of types from Z converging to r', a contradiction.

Let $r_c = \mathrm{stp}(a)$. For any c' realizing a type in X let $r_{c'}$ be the unique image of r_c under any Q-mapping fixing B pointwise and sending c to c'.

X splits into infinitely many $\mathrm{Aut}(Q)$-orbits over Bc. Since by Lemma 3.1 all of them are either finite or open, the open ones are dense in X. So we can choose open $\mathrm{Aut}(Q)$-orbits $X_n \subseteq X, n < \omega$, over Bc, such that they converge (topologically) to $\mathrm{stp}(c)$. Notice that $\mathrm{cl}(X_n) \setminus X_n$, being a nowhere dense set and a union of $\mathrm{Aut}(Q)$-orbits over Bc, must be a union of finite $\mathrm{Aut}(Q)$-orbits over Bc.

Since some neighbourhood of $\mathrm{stp}(c)$ contains no finite $\mathrm{Aut}(Q)$-orbits over Bc (by the claim), discarding some X_n's, we can assume that all the X_n's are clopen in $S(\mathrm{acl}(\emptyset))$.

By the claim, discarding some X_n's, we can assume that for every n, if c_i realize types in X_i for $i < n$ then no type in X_n is Q-finite over $Bc\{c_i, i < n\}$. By the "exchange property" of Q-finiteness this implies that whenever $I \subseteq \omega$ is finite, $j \in \omega \setminus I$ and c_i realizes a type in $X_i, i \in I$, then no type in X_j is Q-finite over $Bc\{c_i, i \in I\}$.

Indeed, in the above notation we have that if $\mathrm{stp}(c_j)$ is Q-finite over $Bc\{c_i, i \in I \cup \{i_0\}\}$ and is not Q-finite over $Bc\{c_i, i \in I\}$, then $\mathrm{stp}(c_{i_0})$ is Q-finite over $Bc\{c_i, i \in I \cup \{j\}\}$.

Suppose that

(a) *if* $n < \omega$ *and* $c_1, \ldots, c_n$ realize types in X, a_i realizes $r_{c_i}, i \leq n$, $\{a_i, c_i, i \leq n\}$ is Bc-independent and c_{n+1} realizes a type in X with open $\mathrm{Aut}(Q)$-orbit over $Bc\{a_i c_i, i \leq n\}$, *then* $r_{c_{n+1}}$ is non-isolated over $\mathrm{acl}(\emptyset) \cup Bc\{a_i c_i, i \leq n\} c_{n+1}$ (equivalently: $r_{c_{n+1}}$ has no forking extension over $Bc\{a_i c_i, i \leq n\} c_{n+1}$).

In this case we shall construct many countable models of T. Let $I \subseteq \omega$. Choose c_i realizing a type in X_i for all $i \in I$. By the omitting types theorem, we can find a model $M \in K_Q$ containing $Bc\{c_i, i \in I\}$, realizing r_{c_i} for $i \in I$ and omitting $r_{c'_j}$ for any $j \in \omega \setminus I$ and any $c'_j \in M$ realizing a type in X_j. This is enough, since we can then recover I from M. This idea is similar to [Bu1]. However, since we are working in K_Q, we cannot apply the omitting types theorem directly.

Specifically, choose a_i realizing $r_{c_i}, i \in I$, with $\{a_i, c_i, i \in I\}$ Bc-independent. Let $B' = Bc\{a_i c_i, i \in I\}$. We find a B'-independent sequence $c'_n, n \in \omega$, such that

(b) c'_n realizes a type in some $X_j, j \in \omega \setminus I$,

(c) there is no c' realizing a type in some $X_j, j \in \omega \setminus I$, with $c' \not\perp B' c'_{\leq n}$ and $r_{c'}$ isolated over $\mathrm{acl}(\emptyset) \cup B' c'_{\leq n}$, and

(d) c'_n is not Q-finite over $B''c'_{<n}$ for any finite $B'' \subseteq B'$.

Suppose we have found $c'_i, i < n$, and want to choose c'_n. Choose any $X_j, j \notin I$, and any c'_n realizing some type in X_j such that (d) holds. If for some c' as in (c), $r_{c'}$ is isolated over $\mathrm{acl}(\emptyset) \cup B'c'_{\leq n}$, then for some finite $B'' \subseteq B'$, $c' \not\perp B''c'_{\leq n}$ and $r_{c'}$ is isolated over $\mathrm{acl}(\emptyset) \cup B''c'_{\leq n}$.

Suppose for a contradiction that $\mathrm{stp}(c')$ is Q-finite over $B''c'_{<n}$. If $c' \not\perp B''c'_{<n}$, then by the inductive hypothesis $r_{c'}$ has no forking extension over $B''c'_{<n}$, but has a forking extension over $B''c'_{\leq n}$. It follows that $r_{c'}$ is Q-finite over $B''c'_{<n}$, hence also c'_n is Q-finite over $B''c'_{<n}$, a contradiction.

If $c' \perp B''c'_{<n}$, then $c' \not\perp c'_n(B''c'_{<n})$ so again we get that c'_n is Q-finite over $B''c'_{<n}$, a contradiction.

Hence $\mathrm{stp}(c')$ is not Q-finite over $B''c'_{<n}$. This gives $c' \not\perp c'_n(B''c'_{<n})$ hence $r_{c'}$ has a forking extension over $B''c'_{<n}c'$. This contradicts (a).

Now let $B^* = B'\{c'_n, n < \omega\}$. We can arrange the choice of $c'_n, n < \omega$, so that $\{\mathrm{stp}(c'_n), n < \omega\}$ is dense in every $X_j, j \notin I$. By the omitting types theorem we can find a model $M \in K_Q$ containing B^* such that for each $j \notin I$, $X_j(M) \subseteq \mathrm{cl}_p(B^*)$ and, for each $c' \in X_j(M)$, M omits $r_{c'}$. Specifically, we omit the following types:

Firstly, for each $j \notin I$, we omit the type $X_j(x) \cup \{$"$x \notin \mathrm{cl}_p(B^*)$"$\}$. Secondly, there are countably many types $r_n \in \bigcup_{j \notin I} X_j$, $n < \omega$, realized in $\mathrm{cl}_p(B^*)$. Each such type r_n determines uniquely a non-isolated type $p_n = \mathrm{tp}(a'c'/Q)$, where c' realizes r_n and a' realizes $r_{c'}$. So we can omit also all the types $p_n, n < \omega$.

But we have assumed that T has few countable models. Hence (a) is false, which means that for some $n < \omega$ we find

(e) $c_1, \ldots, c_n$ realizing types in X, a_i realizing $r_{c_i}, i \leq n$ with $\{a_i, c_i, i \leq n\}$ Bc-independent, and c_{n+1} realizing a type in X with an open $\mathrm{Aut}(Q)$-orbit over $Bc\{a_ic_i, i \leq n\}$ and $r_{c_{n+1}}$ isolated over $\mathrm{acl}(\emptyset) \cup Bc\{a_ic_i, i \leq n\}c_{n+1}$.

This means that for some a_{n+1} realizing $r_{c_{n+1}}$,

$$c_{n+1} \perp Bcc_{\leq n}a_{\leq n} \quad \text{and} \quad a_{n+1}c_{n+1} \not\perp Bcc_{\leq n}a_{\leq n}.$$

Both a_{n+1} and c_{n+1} are not Q-finite over $Bcc_{\leq n}a_{\leq n}$. Now we are in a situation similar as in the proof of [Pi, lemma 2.1]. We can choose n to be minimal possible. This implies that for $i \leq n$, r_{c_i} is not realized in $\mathrm{cl}_p(Bca_{<i}c_{\leq i})$. Enumerate $Bca_{\leq n}c_{\leq n}$ as $\{b_i, i \leq m\}$ so that the elements of B appear first, then c and then $\langle c_j, a_j, j \leq n\rangle$. We can assume (discarding some last b_i's if needed) that $r_{c_{n+1}}$ is not realized in $\mathrm{cl}_p(b_{<m}c_{n+1})$.

Now let $C = Cb(a_{n+1}c_{n+1}/b_{\leq m})$ and choose a finite $b \subseteq C$ with $C \subseteq \operatorname{acl}(b)$ (by superstability). We shall prove that b satisfies our demands with respect to the set $A'a' = Bc_{n+1}a_{n+1}$. Since $Bc_{n+1}a_{n+1}$ and Aa are $\operatorname{Aut}(Q)$-conjugate, this will be enough. Specifically, we will prove that

b is regular, Q-finite, independent of Bc_{n+1} and $a_{n+1} \not\perp b(Bc_{n+1})$.

We need the following fact:

(f) If $d = a_{n+1}c_{n+1}$, $d' \overset{s}{\equiv} d(b_{\leq m})$ and $d' \perp d(b_{\leq m})$, then $d' \not\perp d$.

Suppose for a contradiction that $d \perp d'$. Since $w_p(d/b_{\leq m}) = 1$, we can choose $e \in \operatorname{cl}_p(d) \cap \operatorname{cl}_p(b_{\leq m}) \cap \operatorname{acl}(db_{\leq m})$ with $w_p(e) = 1$ (see [Hr, 4.1]). Choose a long Morley sequence $d_\alpha, \alpha < \omega_1$, in $\operatorname{stp}(d/b_{\leq m})$ and let e_α be a conjugate of e over $d_\alpha b_{\leq m}$.

We see that for $\alpha \neq \beta < \omega_1$, $\operatorname{cl}_p(e_\alpha) \neq \operatorname{cl}_p(e_\beta)$ (since $d_\alpha \perp d_\beta$ and $w_p(d_\alpha d_\beta/e_\alpha e_\beta) = 2$) and $e_\alpha \in \operatorname{cl}_p(b_{\leq m})$. Hence in the cl_p-pregeometry on $p(\mathfrak{C})$, in the cl_p-closure of some finite set there are uncountably many points. On the other hand, the division ring underlying this pregeometry is countable (see Section 1), a contradiction.

Using (f) we get that

(g) $w_p(b) = 1$, $b \in \operatorname{cl}_p(d) \cap \operatorname{acl}(b_{\leq m})$ and $\operatorname{stp}(b)$ is regular (in the ordinary sense).

Indeed, choose $k < \omega$ so large that $b \subseteq \operatorname{dcl}(d_1 \cdots d_k)$. We have $w_p(d_1 \cdots d_k/b) = w_p(d_1 \cdots d_k/b_{\leq m}) = k$. On the other hand,

$$\begin{aligned} w_p(d_1 \cdots d_k/b) + w_p(b) &= w_p(d_1 \cdots d_k b) \\ = w_p(d_1 \cdots d_k) &= w_p(d_2 \cdots d_k/d_1) + w_p(d_1) \leq k+1 \end{aligned}$$

(by (f)). So $w_p(b) \leq 1$. As $w_p(d/b) = 1$, we get that $w_p(b) = 1$ and $b \in \operatorname{cl}_p(d)$.

Since $b \in \operatorname{acl}(b_{\leq m})$ and $b_{\leq m}$ is semiregular, also b is semiregular, and by [Hr], $\operatorname{stp}(b)$ is regular.

Next we prove that

(h) $\operatorname{stp}(b)$ is not realized in $\operatorname{cl}_p(b_{<m})$.

Otherwise, choose $b' \in \operatorname{cl}_p(b_{<m})$ with $b' \overset{s}{\equiv} b$ and then a'_{n+1} with $a'_{n+1}b'c_{n+1} \overset{s}{\equiv} a_{n+1}bc_{n+1}$. We see that $a'_{n+1} \in \operatorname{cl}_p(c_{n+1}b') \subseteq \operatorname{cl}_p(b_{<m}c_{n+1})$, hence $r_{c_{n+1}}$ is realized in $\operatorname{cl}_p(b_{<m}c_{n+1})$, a contradiction.

Similarly, by the minimality of n we get that $\mathrm{stp}(b_m)$ is not realized in $\mathrm{cl}_p(b_{<m})$. Let $\delta(y)$ be a p-formula over $\mathrm{acl}(\emptyset)$, witnessing that b_m is regular, and let $\chi(x, y, \overline{z})$ be a formula, true of $(b, b_m, b_{<m})$, witnessing $b \in \mathrm{acl}(b_{\leq m})$. Since $\mathrm{stp}(b_m)$ is not modular we can assume that $P_\delta = S(\mathrm{acl}(\emptyset)) \cap [\delta]$, and no type in P_δ is modular.

By [Ne4, 2.13], the set of types in P_δ realized in $\mathrm{cl}_p(b_{<m})$ is closed in P_δ, and $\mathrm{stp}(b_m)$ lies outside this set, so diminishing δ we can assume that no type in P_δ is realized in $\mathrm{cl}_p(b_{<m})$. Let $r = \mathrm{stp}(b_{<m})$. Now it is easy to check that

$$\varphi(x) = (\text{“for generic } \overline{z} \text{ realizing } r\text{”}, \exists y(\chi(x, y, \overline{z}) \wedge \delta(y)))$$

is a p-formula true of b, $P_\varphi = [\varphi] \cap S(\mathrm{acl}(\emptyset))$ and $\mathcal{M}(P_\varphi) = 1$.

So by Lemma 3.1 we have that either b is Q-finite or the $\mathrm{Aut}(Q)$-orbit of $\mathrm{stp}(b)$ is open (or, in other words: $\mathrm{tp}(b/Q)$ is Q-isolated). We shall exclude the latter possibility. If $b \not\perp Bc_{n+1}$ then $a_{n+1} \in \mathrm{cl}_p(Bc_{n+1})$, a contradiction. So $b \perp Bc_{n+1}$. If b is not Q-finite and is Q-finite over B, then since $|B| < |A|$, by the induction hypothesis either $\mathrm{stp}(b/B)$ is modular or for some regular Q-finite b' independent of B, $b \not\perp b'$. In the first case we get easily that $\mathrm{stp}(a_{n+1}/Bc_{n+1})$ is modular, so $\mathrm{stp}(a/A)$ is modular, a contradiction. In the second case we satisfy the requirements of the theorem replacing b by b'. So we may assume that b is not Q-finite over B, meaning that $\mathrm{stp}(b/QB)$ is Q-isolated over B.

Then since $\mathrm{tp}(c_{n+1}/b_{\leq m}Q)$ is Q-isolated over $b_{\leq m}$ and $c_{n+1} \perp b_{\leq m}$, we get that $\mathrm{tp}(c_{n+1}b/Q)$ is Q-isolated over B (cf. Lemma 1.1 and [Ne2]).

Hence also $\mathrm{tp}(b/c_{n+1}Q)$ is Q-isolated over Bc_{n+1}. On the other hand, since a_{n+1} is Q-finite over Bc_{n+1} and $b \in \mathrm{cl}_p(a_{n+1}c_{n+1})$, $\mathcal{M}(P_\varphi) = 1$ gives that b is Q-finite over Bc_{n+1}, a contradiction.

So we have proved that b is regular, Q-finite and $a_{n+1} \not\perp_{c_{n+1}} (b)$. This implies also that $a_{n+1} \not\perp_b (Bc_{n+1})$, which finishes the proof.

4. Conclusion

Assume (as in the previous section) that Φ is a countable disjunction of formulas over $\emptyset$ with $\mathrm{acl}(\emptyset) \subseteq \Phi(\mathfrak{C})$, $Q = \Phi(M^*)$ for a countable model M^* of T and p is a meager stationary type over $\emptyset$ orthogonal to Φ.

We use the word “regular” in the sense of the previous section. Let $P = \{\mathrm{stp}(a)\colon a \in \mathfrak{C} \text{ is regular}\}$. For any model M of T let $P^M = \{r \in P\colon r \text{ is realized in } M\}$. Suppose $M \in K_Q$ and $A \subseteq M$ is an independent set of regular elements. We call A a Q-finite basis of M if

(a) every $a \in A$ is Q-finite,

(b) for every Q-finite regular $b \in M$, $\operatorname{stp}(b)$ is realized in $\operatorname{cl}_p(A)$

and A is minimal with respect to (a), (b).

Remark 4.1: For regular b and any $A \subseteq M$, $\operatorname{stp}(b)$ is realized in $\operatorname{cl}_p(A)$ iff $\operatorname{stp}(b)$ is realized in $\operatorname{cl}_p(A) \cap M$.

Proof: Suppose $\operatorname{stp}(b)$ is realized in $\operatorname{cl}_p(A)$. Wlog A is finite. Choose a p-formula $\varphi(x)$ over $\operatorname{acl}(\emptyset)$ witnessing that b is regular. Choose a finite $c \subseteq M$ such that $w_p(c) = 0$ and $U(A/c)$ is minimal possible. $\varphi(x)$ is also a p-formula over c and $\mathcal{M}(P_\varphi) = 1$ (now $P_\varphi = \{r \in S(\operatorname{acl}(c)) \cap [\varphi]\colon w_p(r) = 1\}$).

Also, $\operatorname{stp}(b/c)$ is realized in $\operatorname{cl}_p(A)$, say by b'. Choose a formula $\psi(x)$ over A true of b' such that $\psi(x) \vdash \varphi(x)$ and $\psi(x)$ forks over c. Since $\mathcal{M}(P_\varphi) = 1$ we have that $\operatorname{Tr}_c(\psi(x)) \cap P_\varphi$ is finite. Wlog $\operatorname{Tr}_c(\psi(x)) \cap P_\varphi = \{\operatorname{stp}(b/c)\}$.

Let $b'' \in M$ realize $\psi(x)$. If $\operatorname{stp}(b''/c) \notin P_\varphi$ then $w_p(b''c) = 0$ and $U(A/b''c) < U(A/c)$, contradicting the choice of c. So $\operatorname{stp}(b''/c) \in P_\varphi$ and $\operatorname{stp}(b''/c) = \operatorname{stp}(b/c)$. Hence $\operatorname{stp}(b)$ is realized in $\operatorname{cl}_p(A) \cap M$.

Theorem 2.6 implies the following.

LEMMA 4.2: *If $M \in K_Q$ is countable, then M contains a finite Q-finite basis and all Q-finite bases of M have the same size.*

We shall prove the following:

THEOREM 4.3: *Assume $M \in K_Q$ is countable and A is a Q-finite basis of M. Then there is a countable $M' \in K_Q$ Q-atomic over A such that $P^M = P^{M'}$. In particular, up to an isomorphism preserving Q setwise, there are countably many sets in $\{P^M\colon M \in K_Q$ is countable$\}$.*

Proof: By Lemma 4.2, A is finite. By Lemma 1.1 we find a countable $N \in K_Q$ Q-atomic over A. Notice that A is a Q-finite basis of N. Let $B \subseteq M$ [$B' \subseteq N$, respectively] be a maximal set of regular elements such that

(a) $A \cup B$ [$A \cup B'$, respectively] is independent,

(b) B [B', respectively] is Q-atomic over A, and

(c) for every p-formula φ over $\operatorname{acl}(\emptyset)$ with $\mathcal{M}(P_\varphi) = 1$, $\{\operatorname{stp}(b)\colon b \in B\}$ [$\{\operatorname{stp}(b)\colon b \in B'\}$, respectively] is dense in P_φ.

In order to satisfy (c) we build B in countably many steps. At each step a larger and larger finite part B_0 of B is built so that (a) and (b) are satisfied. Then we consider some φ as in (c) and some open set U in P_φ. By Lemma 3.1

we may find a $b \in M$ such that (a) and (b) still hold for $B_0 \cup \{b\}$ and $\mathrm{stp}(b) \in U$. Since there are countably many φ as in (c), in countably many steps we can ensure that (c) holds. Similarly we deal with B'.

We shall prove that

(d) every $r \in P^M$ $[P^N$, respectively$]$ is realized in $\mathrm{cl}_p(AB)$ $[\mathrm{cl}_p(AB')$, respectively$]$.

We deal with M only. Suppose $a \in M$ is regular and $\mathrm{stp}(a)$ is not realized in $\mathrm{cl}_p(AB)$. By the maximality of B, for some finite $B'' \subseteq B$, $\mathrm{tp}(aB''/QA)$ is not Q-isolated over A. By Lemma 1.1, $\mathrm{tp}(a/QAB'')$ is not Q-isolated over AB''. By Lemma 3.1, a is Q-finite over AB''. a is not Q-finite over $\emptyset$ (otherwise $\mathrm{stp}(a)$ would be realized in $\mathrm{cl}_p(A)$, since A is a Q-finite basis of M). Hence also $\mathrm{stp}(a)$ is not modular. By Theorem 3.2 we have two cases.

CASE 1: *$\mathrm{stp}(a/AB'')$ is modular.* Then $\mathrm{stp}(a)$ is realized in $\mathrm{cl}_p(AB'')$.

CASE 2: *$a \not\perp b(AB'')$ for some regular Q-finite b independent of AB''.* Again, we get that $\mathrm{stp}(b)$ is realized in M. Since A is a Q-finite basis, $\mathrm{stp}(b)$ is realized in $\mathrm{cl}_p(A)$. This also implies that $\mathrm{stp}(a)$ is realized in $\mathrm{cl}_p(AB'')$.

As in the proof of Lemma 1.1(5)(see [Ne2]), using (c) we can find $f \in \mathrm{Aut}(Q/A)$ with $f(B') = B$. Let $M' = f(N)$. So $r \in P^M$ iff r is realized in $\mathrm{cl}_p(AB)$ iff $r \in P^{M'}$, hence $P^M = P^{M'}$.

For the last clause notice that P^M is determined up to isomorphism by $\mathrm{tp}(A/Q)$, or even by the pseudo-type of A over Q. Since A is an independent set of Q-finite regular elements, $\mathrm{tp}(A/Q)$ is τ-stable (see [Ne3]). By [Ne3] there are countably many good τ-stable pseudo-types over Q, hence we are done.

Now we assume that G is a 0-definable meager group, $\mathcal{M}(G) = 1$ and p is the generic type of G^0. In this case we set $\Phi(x)$ so that $\Phi(\mathfrak{C}) = \mathrm{acl}(\emptyset) \cup G^-$, where $G^- = \mathrm{cl}_p(\emptyset) \cap G$. As before, $Q = \Phi(M^*)$ for some countable model M^* of T and we try to classify sets of the form $G(M)$, where $M \in K_Q$ is countable. Notice that here for an element a of G, a is generic iff a is regular. By Theorem 4.2 we can describe $\mathcal{G}^M$, that is, we know which cosets of $G^0(M)$ are realized in M.

Below we shall see that we can choose a cl_p-basis C of $G(M)$ extending a Q-finite basis A of M, so that the pseudo-type of C over Q is determined by the pseudo-type of A over Q and $\dim(p(M))$. Since we know which cosets of $G^0(M)$ are realized in M (namely these containing a generic element in $\mathrm{cl}_p(C)$ or meeting $G^-(M)$), in order to describe $G(M)$ it is enough to

(*) describe $G^0(M)$ and the possible ways in which a generic element from $\mathrm{cl}_p(C)$ may be chosen in a given coset of $G^0(M)$.

We may choose C so that it contains a basis C_0 of $p(M)$. We know that $G^0(M)$ is essentially a vector space over F_G. So again in order to describe $G^0(M)$ over C and Q it is enough to

(**) describe the possible ways in which a generic element from $\mathrm{cl}_p(C_0) \cap G^0(M)$ may be chosen in a given coset of $G^-(M)$.

To render (*) and (**), [Bu1] and [Pi] use some additional properties of G: weak minimality or the (weaker) assumption that any forking extension of a generic type in G has Morley rank. These facts imply that we can choose representatives of the cosets in question in the prime model of $\mathrm{Th}(G)$ over QC, contained in M, and then clearly $G(M)$ must be prime (that is, atomic) over QC. This completely describes the isomorphism type of M in these cases.

In general we cannot hope for that much. The possibility to choose in some canonical way representatives of the cosets mentioned above seems to depend on the properties of the pseudo-endomorphisms of G. For instance, suppose $c \in G^0 \cap \mathrm{cl}_p(C_0) \setminus G^-$. Then c, or more properly $c + G^-$, is in the span (with respect to the ring F_G) of C_0. This tells us that $c + G^-$ meets M, but this does not necessarily determine the isomorphism type of $(c+G^-)(M)$ over $C_0 \cup G^-(M)$. This is so because the images of elements of G^0 via pseudo-endomorphisms are determined only up to G^-, so we do not have in general a method of picking an element in $(c + G^-) \cap M$ in a canonical way.

If the ring of pseudo-endomorphisms of G is just a prime field (necessarily finite), we have such a method and can prove Vaught's conjecture for $\mathrm{Th}(G)$ relative to Φ (in fact, relative to G^-) and some finite tuple realizing an isolated type over $\emptyset$.

Assume M is any model of T. We can choose generic $e_0, \dots, e_k \in G(M)$ with the following properties:

1. $\mathrm{tp}(e_0 \dots e_k)$ is isolated,
2. $e_0, \dots, e_k$ are pairwise dependent, and
3. $\mathrm{stp}(e_0), \dots, \mathrm{stp}(e_k)$ are pairwise distinct and

$$\mathrm{CL}(e_0) \cap (\mathcal{G} \setminus \mathcal{G}m) = \{\mathrm{stp}(e_0), \dots, \mathrm{stp}(e_k)\}$$

(that is, whenever $e \in G \setminus Gm$ is generic and depends on e_0, then $\mathrm{stp}(e) = \mathrm{stp}(e_i)$ for some i).

Here CL is defined as in the beginning of Section 2, for $\varphi = G$. Clearly, for every i, $\mathrm{stp}(e_i)$ is non-modular. Let $E = \{e_0, \dots, e_k\}$. Assuming F_G is a prime field, we shall prove Vaught's conjecture for G relative to G^- and E. We shall use the following lemma:

LEMMA 4.4: *Assume F_G is a prime field. Then there are open neighbourhoods V_i of $\mathrm{stp}(e_i), i \le k$, in $S(\mathrm{acl}(\emptyset))$ and numbers $n_{ij} > 0$, with $n_{ii} = 1$, such that for every i, if e'_i realizes a generic type in V_i and $e'_j = e_j + n_{ij}(e'_i - e_i), j \le k$, then the elements $e'_j, j \le k$ satisfy conditions (2), (3) above.*

Proof: The proof is similar to [Lo], [Ne6]. Choose $e'_0 \cdots e'_k \overset{s}{\equiv} e_0 \cdots e_k$ with $e'_0 \cdots e'_k \mathop{\smile\!\!\!\!\!\!\;|}\, e_0 \cdots e_k$. Let $c_i = e'_i - e_i$. We see that $c_0, \ldots, c_k$ are generic in G^0, and are pairwise dependent. Since F_G is a prime field, there are numbers $n_{ij} > 0$ such that $c_i - n_{ji}c_j \in G^-$. We have that $n_{ij} \cdot n_{ji} \equiv 1 \pmod q$, where $q = \mathrm{char}(F_G)$. As in [Lo], [Ne6] it follows that for every i, if $e^*_i \overset{s}{\equiv} e_i$ then letting $e^*_j = e_j + n_{ij}(e^*_i - e_i)$ we have that

(a) $e^*_0, \ldots, e^*_k$ satisfy (2) and (3).

(a) is clear when $e^*_i \mathop{\smile\!\!\!\!\!\!\;|}\, e_i$. Now suppose e^*_i is arbitrary with $e^*_i \overset{s}{\equiv} e_i$. Wlog $e'_i \mathop{\smile\!\!\!\!\!\!\;|}\, e^*_i e_i$. Let

$$b_j = e'_j + n_{ij}(e_i - e'_i) \quad \text{and} \quad b'_j = e'_j + n_{ij}(e^*_i - e'_i).$$

We see that $b'_j = b_j + n_{ij}(e^*_i - e_i)$. By (a) (for the case when $e^*_i \mathop{\smile\!\!\!\!\!\!\;|}\, e_i$, applied to (e^*_i, e'_i) and (e_i, e'_i)) we get that $b_j \not\mathop{\smile\!\!\!\!\!\!\;|}\, e_i$, $b'_j \not\mathop{\smile\!\!\!\!\!\!\;|}\, e^*_i$ and $b_j - e_j \in G^-$. Hence also $b'_j - e^*_j \in G^-$, so $e^*_j \not\mathop{\smile\!\!\!\!\!\!\;|}\, e^*_i$. This proves (a).

By compactness, (a) holds also for every generic e^*_i realizing a type in some open neighbourhood V_i of $\mathrm{stp}(e_i)$. This finishes the proof.

THEOREM 4.5: *Assume the ring of pseudo-endomorphisms of G is a prime field, $M \in K_Q$ is a countable model of T and, for every $n < \omega$, there are countably many good pseudo-types in $S_n(Q)$. Then there is a finite set $D \subseteq M$ and a basis C of $p(M)$ over D such that $G(M)$ is Q-atomic over DC. Hence Vaught's conjecture is true for K_Q.*

Proof: Wlog $E \subseteq M$. Let A be a Q-finite basis of M containing a realization of p if $p(M) \neq \emptyset$. Since the pre-geometry is locally finite, we can choose a finite set A' of generic elements of $G(M)$ such that $A' \subseteq \mathrm{cl}_p(A)$ and, if a G^--coset meets $\mathrm{cl}_p(A) \cap M$, then it meets A'.

As in Lemma 4.3, we can choose a maximal set $B \subseteq G(M)$ of regular elements such that $e_0 \in B$, AB is independent, B is Q-atomic over A and $\{\mathrm{stp}(b) \colon b \in B\}$ is dense in $\mathcal{G}$. As in the proof of Lemma 4.3 we have

(a) every $r \in \mathcal{G}^M$ is realized in $\mathrm{cl}_p(AB)$.

The proof of Lemma 4.3 goes through even though B is now chosen only to be maximal in $G(M)$ (not in M^{eq}).

CLAIM: *Suppose B is Q-atomic over A and $a \in \mathrm{cl}_p(A) \cap M$. Then there is a finite $B_0 \subseteq B$ such that B is Q-atomic over AaB_0.*

Proof: We work in $T(Aa)$. Let $B = \{b_n, n < \omega\}$, $r_n = \mathrm{stp}(b_n)$. Suppose b_n is not Q-atomic over $b_{<n}$. By Theorem 3.4 either b_n is Q-finite over $\emptyset$ or $\mathrm{stp}(b_n/b_{<n})$ is modular, or for some regular Q-finite c_n independent of $b_{<n}$, $b_n \not\perp c_n (b_{<n})$.

There are at most finitely many n for which the first case holds (by Proposition 2.2). The second case is impossible, since in this case b_n would be Q-finite over $Ab_{<n}$ (in the original signature), hence $\mathrm{tp}(b_n/Ab_{<n})$ would not be Q-isolated over $Ab_{<n}$. Also, using the basis lemma one gets that the third case may happen only finitely many times.

Indeed, let $I = \{n$: the third case occurs$\}$ and $S = \{c_n : n \in I\}$. Choose finitely many increasing $n_1, \ldots, n_k \in I$ such that for every $c \in S$, $\mathrm{stp}(c)$ is realized in $\mathrm{cl}_p(c_{n_1}, \ldots, c_{n_k})$. We see that for every $n \in I$ larger than $\max\{n_1, \ldots, n_k\}$, $\mathrm{stp}(b_n/b_{<n})$ is modular. So I must be finite. This proves the claim.

Clearly, B is Q-atomic over Ae_0. Using the claim we find a finite $B' \subseteq B$ such that B is Q-atomic over $AA'B'E$. Let $D = AA'B'E$.

Let C be a basis of $p(M)$ over D. Since $\mathcal{M}(p) = 0$, no element of C is Q-atomic over D. Hence BCD is independent. Also, since $\mathrm{Aut}(Q/D) = \mathrm{Aut}(Q/CD)$ (regarded as a group of permutations of Q), we have

(b) B is Q-atomic over CD.

It is enough to prove that

(c) $G(M) \subseteq \mathrm{dcl}(BCDQ)$.

Since F_G is a prime field, every pseudo-endomorphism of G is represented by a function f_n mapping x to nx, $n \in F_G$, the functions f_n are defined on all of G. It follows that for every $X \subseteq G$ and $x \in G$, $x \in \mathrm{cl}_p(AX)$ iff $x \in \mathrm{cl}_p(A'X)$.

Hence every co-set of $G^-(M)$ in $G^0(M)$ contains an element in $\mathrm{dcl}(A'C)$, which gives $G^0(M) \subseteq \mathrm{dcl}(A'CQ)$. So to prove (c) it is enough to show that every co-set of $G^0(M)$ in $G(M)$ contains an element in $\mathrm{dcl}(BCDQ)$. Since $\{\mathrm{stp}(b) : b \in B\}$ is dense in $\mathcal{G}$, it suffices to show that for some open non-empty set V_0' in $\mathcal{G}$, every type in $V_0' \cap \mathcal{G}^M$ is realized in $\mathrm{dcl}(BCDQ)$.

Let V_0 be the set from Lemma 4.4 and choose an open set $V_0' \subseteq V_0$ containing $\mathrm{stp}(e_0)$ such that whenever e_0' realizes a type in $V_0' \cap \mathcal{G}$, then $\mathrm{CL}(e_0') \subseteq \bigcup_i V_i \cup \mathcal{G}m$.

Suppose $r \in V_0' \cap \mathcal{G}^M$. By (a), r is realized in $\mathrm{cl}_p(AB)$. So choose b realizing r with $b \in \mathrm{cl}_p(AB)$ (b may lie outside M!). However, since the functions $f_n, n \in F_G$, are defined on all of G, there is some b' in the $\{f_n, n \in F_G\}$-span of $A'B$ such that $b \not\perp b'$. The $\{f_n, n \in F_G\}$-span of $A'B$ is simply the subgroup of G generated by $A'B$, so $b' \in \mathrm{dcl}(A'B)$, whence $b' \in M$. b' realizes a generic type in some V_i.

Let $b'' = e_0 + n_{0i}(b' - e_i)$. So $b'' \in \mathrm{dcl}(A'BE)$. According to Lemma 4.4, $b'' \not\perp b$ and $b'' \stackrel{s}{\equiv} b$. So we have (c).

Corollary 4.6: *Assume G is a meager group of U-rank ω and $\mathcal{M}$-rank 1, and F_G is a prime field. Then Vaught's conjecture holds for* $\mathrm{Th}(G)$.

Proof: Let $T = \mathrm{Th}(G)$. $T \lceil G^-$ is a superstable many sorted theory of finite U-rank, so we have Vaught's conjecture for it by [Bu2]. Moreover, Buechler proves there that any countable model of such a theory is prime over a Morley sequence. Hence by [Ne3, Ne4], for every countable model M, all good pseudo-types over $Q = G^-(M)$ are τ-stable, hence there are countably many of them. Applying Theorem 4.5 we get Vaught's conjecture for T relative to Q. Since we have Vaught's conjecture for $T \lceil G^-$, we have it for T.

Actually, checking the proof one can see that in Corollary 4.6, if $\mathrm{Th}(G)$ has few countable models, then $G(M)$ is just atomic over some finite set E and finitely many Morley sequences over E.

An important point in the proof of Theorem 4.5 was the assumption that F_G is a prime field. This assumption implies that we can represent elements of F_G as real endomorphisms of G, with values defined firmly in G (and not only up to G^-). We did not use the finiteness of F_G for anything else.

Now suppose F_G is finite (and not necessarily a prime field). Then we can find a single 0-definable non-generic subgroup H of G, a 0-definable generic subgroup G' of G and representatives of elements of F_G which are defined on G', such that these representatives, and also compositions of any two of them, have values determined up to H (or, in other words, belonging to G/H). In this situation we have the following corollary:

Corollary 4.7: *Assume G is a meager group of $\mathcal{M}$-rank 1, with F_G finite,* $\mathrm{Th}(G)$ *has few countable models and H is a subgroup of G described above. Let $G^* = G/H$. If there are countably many good pseudo-types over any countable $Q = (G^*)^-(M)$, then Vaught's conjecture is true for G^* relative to $(G^*)^-$.*

Proof: We choose E as in Theorem 4.5, and add to it representatives of G'-cosets of G. Then we choose ABC in $G'(M)$ like in the proof of Theorem 4.5

(for $G := G'$) (A is chosen as a Q-finite basis of G' over $(G^*)^-$). Then using the above remarks we show that $G^*(M) \subseteq \mathrm{acl}(CD(G^*)^-(M))$.

COROLLARY 4.8: *Assume G is a meager group of U-rank ω and $\mathcal{M}$-rank 1, with F_G finite, and* $\mathrm{Th}(G)$ *has few countable models. Then for some non-generic 0-definable $H \leq G^0$,* $\mathrm{Th}(G/H)$ *has countably many countable models.*

References

[Ba] J. T. Baldwin, *Fundamentals of Stability Theory*, Springer, Berlin, 1988.

[Bu1] S. Buechler, *Classification of small w.m. sets, I*, in *Classification Theory, Proceedings, Chicago 1985* (J. T. Baldwin, ed.), Springer, Berlin, 1987, pp. 32–71.

[Bu2] S. Buechler, *Vaught's conjecture for superstable theories of finite rank*, Annals of Pure and Applied Logic, to appear.

[Hr] E. Hrushovski, *Locally modular regular types*, in *Classification Theory, Proceedings, Chicago 1985* (J. T. Baldwin, ed.), Springer, Berlin, 1987, pp. 132–164.

[HP] E. Hrushovski and A. Pillay, *Weakly normal groups*, in *Logic Colloquium 1985* (The Paris Logic Group, ed.), North-Holland, Amsterdam, 1987, pp. 233–244.

[HS] E. Hrushovski and S. Shelah, *A dichotomy theorem for regular types*, Annals of Pure and Applied Logic **45** (1989), 157–169.

[Lo] J. Loveys, *Abelian groups with modular generics*, Journal of Symbolic Logic **56** (1991), 250–259.

[M] M. Makkai, *A survey of basic stability theory, with particular emphasis on orthogonality and regular types*, Israel Journal of Mathematics **49** (1984), 181–238.

[Ne1] L. Newelski, *A proof of Saffe's conjecture*, Fundamenta Mathematicae **134** (1990), 143–155.

[Ne2] L. Newelski, *A model and its subset*, Journal of Symbolic Logic **57** (1992), 644–658.

[Ne3] L. Newelski, *Scott analysis of pseudo-types*, Journal of Symbolic Logic **58** (1993), 648–663.

[Ne4] L. Newelski, *Meager forking*, Annals of Pure and Applied Logic **70** (1994), 141–175.

[Ne5] L. Newelski, *$\mathcal{M}$-rank and meager types*, Fundamenta Mathematicae **146** (1995), 121–139.

[Ne6] L. Newelski, *$\mathcal{M}$-rank and meager groups*, Fundamenta Mathematicae **150** (1996), 149–171.

[Ne7] L. Newelski, *$\mathcal{M}$-gap conjecture and m-normal theories*, Israel Journal of Mathematics **106** (1998), 285–311.

[Pi] A. Pillay, *Certain locally modular regular superstable groups*, preprint 1992.

[SHM] S. Shelah, L. Harrington and M. Makkai, *A proof of Vaught's conjecture for $\aleph_0$-stable theories*, Israel Journal of Mathematics **49** (1984), 259–278.

[Sh] S. Shelah, *Classification Theory*, 2nd edn., North-Holland, Amsterdam, 1990.